Module - 01 [Question Bank]

① find the angle between normal to the surface $xy = z^2$ at the points $(4,1,2)$ and $(3,3,-3)$

Given: $\phi = xy - z^2$ —①

We know that, the angle b/w two surfaces is the angle b/w the normals of the two surfaces

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi_1|}$$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = y\hat{i} + x\hat{j} + (-2z)\hat{k}$$

$$\nabla\phi = y\hat{i} + x\hat{j} - 2z\hat{k}$$

$$\nabla\phi_{(4,1,2)} = \hat{i} + 4\hat{j} - 4\hat{k} \longrightarrow (A)$$

$$|\nabla\phi| = \sqrt{1^2+4^2+(-4)^2} = \sqrt{33} \rightarrow |A|$$

similarly at $(3,+3,-3)$

$$\nabla\phi_{(3,3,-3)} = 3\hat{i} + 3\hat{j} + 6\hat{k} \longrightarrow (B)$$

$$|\nabla\phi| = \sqrt{3^2+3^2+6^2} = \sqrt{54} \rightarrow |B|$$

Now, the angle b/w two surface is given that

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} \Rightarrow \frac{A \cdot B}{|A|\,|B|}$$

$$\cos\theta = \frac{(\hat{i}+4\hat{j}-4\hat{k})\cdot(3\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{33}\sqrt{54}}$$

$$\cos\theta = \frac{3+12-24}{9\sqrt{22}} = \frac{-\cancel{9}}{\cancel{9}\sqrt{22}} = \frac{-1}{\sqrt{22}}$$

② find the angle between surfaces $x^2+y^2+z^2 = 9$ and $x^2+y^2-3 = z$ at the point $(2,-1,2)$

Given: $\phi_1 = x^2+y^2+z^2-9$

$\phi_2 = x^2+y^2-z-3$

WKT, The angle b/w two surface is $\cos\theta = \dfrac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$(\nabla\phi_1)_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\nabla\phi_1 = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla\phi_{2(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$$

$$|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21}$$

$$\cos\theta = \frac{4\hat{i} - 2\hat{j} + 4\hat{k} \cdot 4\hat{i} - 2\hat{j} - \hat{k}}{6\sqrt{21}} = \frac{16+4-4}{6\sqrt{21}} = \frac{\cancel{16}^{\,8}}{\cancel{6}_{3}\sqrt{21}}$$

$$\boxed{\cos\theta = \frac{8}{3\sqrt{21}}}$$

(3) If $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$ find $\text{div}\vec{F}$ and $\text{Curl}\vec{F}$

$$\nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$\nabla\phi = (3x^2-3yz)\hat{i} + (3y^2-3xz)\hat{j} + (3z^2-3xy)\hat{k}$$

$$\text{div.}\vec{F} = \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$$

$$\text{div.}\vec{F} = 6x + 6y + 6z$$

$$\boxed{\text{div.}\vec{F} = 6(x+y+z)}$$

$$\text{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(3z^2-3xy) - \frac{\partial}{\partial z}(3y^2-3xz)\right) - \hat{j}\left(\frac{\partial}{\partial x}(3z^2-3xy) - \frac{\partial}{\partial z}(3x^2-3yz)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(3y^2-3xz) - \frac{\partial}{\partial y}(3x^2-3yz)\right)$$

$$= \hat{i}(-\cancel{3x}+\cancel{3x}) - \hat{j}(-\cancel{3y}+\cancel{3y}) + \hat{k}(-\cancel{3z}+\cancel{3z})$$

$$= \underline{\underline{0}}$$

④ If $\vec{F} = \nabla(xy^3z^2)$ find $\operatorname{div}\vec{F}$ and $\operatorname{Curl}\vec{F}$ at the point $(1,-1,1)$

$$\nabla\phi = \frac{\partial}{\partial x}(xy^3z^2)\hat{i} + \frac{\partial}{\partial y}(xy^3z^2)\hat{j} + \frac{\partial}{\partial z}(xy^3z^2)\hat{k}$$

$$\nabla\phi = (y^3z^2)\hat{i} + (3xy^2z^2)\hat{j} + (2xy^3z)\hat{k}$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(y^3z^2) + \frac{\partial}{\partial y}(3xy^2z^2) + \frac{\partial}{\partial z}(2xy^3z)$$

$$= 0 + 6xyz^2 + 2xy^3$$

$$\operatorname{div}\vec{F}_{(1,-1,1)} = 6(-1) + 2(-1)$$

$$= -6 - 2 = \underline{\underline{-8}}$$

$$\operatorname{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(2xy^3z) - \frac{\partial}{\partial z}(3xy^2z^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(2xy^3z) - \frac{\partial}{\partial z}(y^3z^2)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}(3xy^2z^2) - \frac{\partial}{\partial y}(y^3z^2)\right)$$

$$= \hat{i}\left(\cancel{6xy^2z} - \cancel{6xy^2z}\right) - \hat{j}\left(\cancel{2y^3z} - \cancel{2y^3z}\right) + \hat{k}\left(\cancel{3y^2z^2} - \cancel{3y^2z^2}\right)$$

$$\underline{\underline{\operatorname{Curl}\vec{F} = 0}}$$

⑤ Find the Directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at $P(2,-1,2)$ along $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Given :

$$\phi = 4xz^3 - 3x^2y^2z$$

WKT, Directional Derivative formula

$$D.D = \nabla\phi \cdot \frac{\vec{A}}{|\vec{A}|}$$

Now,

$$\nabla\phi = \frac{\partial}{\partial x}(4xz^3 - 3x^2y^2z)\hat{i} + \frac{\partial}{\partial y}(4xz^3 - 3x^2y^2z)\hat{j} + \frac{\partial}{\partial z}(4xz^3 - 3x^2y^2z)\hat{k}$$

$$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 3x^2y^2)\hat{k}$$

$$\nabla\phi_{(2,-1,2)} = 32 - 24\hat{i} + 48\hat{j} + 96 - 12\hat{k}$$

$$\nabla\phi_{(2,-1,2)} = 8\hat{i} + 48\hat{j} + 84\hat{k}$$

$$|A| = \sqrt{2^2+(-3)^2+6^2} = \sqrt{4+9+36} = \sqrt{49} = 7.$$

$$D.D = \frac{8\hat{i}+48\hat{j}+84\hat{k} \cdot 2\hat{i} - 3\hat{j}+6\hat{k}}{7}$$

$$D.D = \frac{16-144+504}{7} = \boxed{\frac{376}{7}}$$

⑥ Show that $\vec{F} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2}$ is both solenoidal and irrotational.

Given: $\vec{F} = \frac{x\hat{i}+y\hat{j}}{x^2+y^2} \Rightarrow \frac{x}{x^2+y^2}\hat{i} + \frac{y}{x^2+y^2}\hat{j}$

W.K.T If Div $\vec{F} = 0$ then it is solenoidal
If Curl $\vec{F} = 0$ then it is irrotational.

$$\text{Div}\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$$

[∵ Applying quotient rule]

$$= \frac{x^2+y^2(1) - x.2x}{(x^2+y^2)^2} + \frac{x^2+y^2(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\text{Div}\vec{F} = \frac{\cancel{2x^2}+\cancel{2y^2}-\cancel{2x^2}-\cancel{2y^2}}{(x^2+y^2)^2} = \underline{0}$$ Hence it is solenoidal

$$\text{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \hat{i}\left(-\cancel{\frac{\partial}{\partial z}\left(\frac{y}{x^2+y^2}\right)}^{0}\right) - \hat{j}\left(-\cancel{\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)}^{0}\right) + \hat{k}\left(\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right)$$

$$= \hat{k}\left(\frac{x^2+y^2(0) - y(2x)}{(x^2+y^2)^2} - \frac{x^2+y^2(0) - x(2y)}{(x^2+y^2)^2}\right)$$

$$\text{Curl}\vec{F} = \hat{k}\left(\frac{-\cancel{2xy}+\cancel{2xy}}{(x^2+y^2)^2}\right) = 0$$ Hence it is irrotational

(7) find the constants a, b & c such that $\vec{F} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (x+cy+2z)\hat{k}$ is a an irrotational.

Given: $\vec{F} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (x+cy+2z)\hat{k}$

WKT curl $\vec{F} = 0$ then it is irrotational.

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(x+cy+2z) - \frac{\partial}{\partial z}(bx+2y-z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(x+cy+2z) - \frac{\partial}{\partial z}(x+y+az)\right) + \hat{k}\left(\frac{\partial}{\partial x}(bx+2y-z) - \frac{\partial}{\partial y}(x+y+az)\right)$$

$$= \hat{i}(c+1) - \hat{j}(1-a) + \hat{k}(b-1)$$

curl $\vec{F}$ -

$c+1=0$ $\quad$ $1-a=0$ $\quad$ $b-1=0$

$\boxed{c=-1}$ $\quad$ $\boxed{a=1}$ $\quad$ $\boxed{b=1}$

$\therefore$ The constants a, b and c are 1, 1 & -1 respectively.

(9) find the directional derivative of $\phi = xy^2 + yz^3$ at $(1,2,-1)$ in the direction normal to the surface $x\log z - y^2 = -4$ at $(-1,2,1)$.

Given: $\phi = xy^2 + yz^3$

$x\log z - y^2 = -4 \Rightarrow f = x\log z - y^2 + 4$

W.KT, $\nabla\phi = \frac{\partial}{\partial x}(\phi)\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$\nabla\phi = (y^2)\hat{i} + (2xy + z^3)\hat{j} + (3yz^2)\hat{k}$

$\nabla\phi_{(1,2,-1)} = 4\hat{i} + 3\hat{j} + 6\hat{k}$

$\nabla f = \frac{\partial}{\partial x}(x\log z - y^2 + 4)\hat{i} + \frac{\partial}{\partial y}(x\log z - y^2 + 4)\hat{j} + \frac{\partial}{\partial z}(x\log z - y^2 + 4)\hat{k}$

$= \log z\,\hat{i} + (-2y)\hat{j} + \frac{x}{z}\hat{k}$

$\nabla f_{(-1,2,1)} = 0\hat{i} + (-4)\hat{j} + -1\hat{k} \Rightarrow \boxed{-4\hat{j} - \hat{k}}$

Now,

$$D.D = \nabla\phi \cdot \frac{\nabla F}{|\nabla F|}$$

$$D.D = \frac{4\hat{i} + 3\hat{j} + 6\hat{k} \cdot -4\hat{j} - \hat{k}}{\sqrt{16+1}} = \frac{-12-6}{\sqrt{17}} = \frac{-18}{\sqrt{17}}$$

(10) find the constant 'a' so that $\vec{F} = y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$ is solinoidal.

Given: $\vec{F} = y(ax^2+z)\hat{i} + x(y^2-z^2)\hat{j} + 2xy(z-xy)\hat{k}$

If $\text{Div}\vec{F} = 0$ then it is solinoidal.

$$\text{Div}\vec{F} = \frac{\partial}{\partial x} y.(a.x^2+z) + \frac{\partial}{\partial y} x(y^2-z^2) + \frac{\partial}{\partial z} 2xy(z-xy)$$

$$\text{Div}\vec{F} = a.2xy + 2xy + 2xy$$

$$\text{Div}F = 2axy + 4xy$$

$$2axy = -4xy$$

$$\boxed{a = -2}$$

(11) If $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$ show that $\vec{F}.\text{curl}\vec{F} = 0$

Given: $F = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$

$$\text{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y+1) & 1 & -x-y \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial z}(x+y+1)\right) + \hat{k}\left(\frac{\partial(1)}{\partial x} - \frac{\partial}{\partial y}(x+y+1)\right)$$

$$= \hat{i}(-1) - \hat{j}(-1) + \hat{k}(-1)$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$\vec{F}.\text{curl}\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k} \cdot -\hat{i} + \hat{j} - \hat{k}$$

$$= -(x+y+1) + 1 + x + y$$

$$= \cancel{-x} \cancel{-y} \cancel{-1} \cancel{+1} \cancel{+x} \cancel{+y}$$

$= \underline{0}$ Hence proved.

(15) Find the constants a and b such that $\vec{F} = (axy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (bxz^2 - y)\hat{k}$ is irrotational

wkt, if $\text{curl}\,\vec{F} = 0$ then it is said to be irrotational

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy+z^3 & 3x^2-z & (bxz^2-y) \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(bxz^2-y) - \frac{\partial}{\partial z}(3x^2-z)\right) - \hat{j}\left(\frac{\partial}{\partial x}(bxz^2-y) - \frac{\partial}{\partial z}(axy+z^3)\right)$$
$$\hat{k}\left(\frac{\partial}{\partial x}(3x^2-z) - \frac{\partial}{\partial y}(axy+z^3)\right)$$

$$\text{Curl}\,\vec{F} = \hat{i}(0-\cancel{1}) - (0-\cancel{1}) - \hat{j}(bz^2 - 3z^2) + \hat{k}(6x - ax)$$

$$bz^2 - 3z^2 = 0 \qquad\qquad 6x - ax = 0$$
$$b\cancel{z^2} = 3\cancel{z^2} \qquad\qquad 6\cancel{x} = a\cancel{x}$$
$$\boxed{b = 3} \qquad\qquad \boxed{a = 6}$$

$\therefore$ The constants a and b are 6 & 3 respectively.

(13) Analyse the constant a such that $\vec{F} = (axy - z^3)\hat{i}$ $\cancel{+(3x^2-z)\hat{j}+(bxz^2-y)\hat{k}}$ ~~is~~ $+ (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is irrotational.

Given: $\vec{F} = (axy - z^3)\hat{i} + (ax^2 - 2x^2)\hat{j} + (xz^2 - axz^2)\hat{k}$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & ax^2 - 2x^2 & xz^2 - axz^2 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(xz^2 - axz^2) - \frac{\partial}{\partial z}(ax^2 - 2x^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(xz^2 - axz^2)\right.$$
$$\left. - \frac{\partial}{\partial z}(axy - z^3)\right) + \hat{k}\left(\frac{\partial}{\partial x}(ax^2 - 2x^2) - \frac{\partial}{\partial y}(axy - z^3)\right)$$

$$= \hat{i}(0) - \hat{j}(z^2 - az^2 + 3z^2) + \hat{k}(2ax - 4x - ax)$$

$$= 4z^2 - az^2 = 0 \qquad\qquad a\cancel{x} = 4\cancel{x}$$
$$\boxed{a = 4} \qquad\qquad \boxed{a = 4}$$

(14) Express vector $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in cylindrical Coordinate System.

Given: $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$

Let $\vec{F} = F_\rho \hat{T}_\rho + F_\phi \hat{T}_\phi + F_z \hat{T}_z$ be the cylindrical coordinate system — (1)

where $F_\rho = \vec{F}\cdot\hat{T}_\rho$, $F_\phi = \vec{F}\cdot\hat{T}_\phi$, $F_z = \vec{F}\cdot\hat{T}_z$

$$\boxed{\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}}$$

consider $F_\rho = \vec{F}\cdot\hat{T}_\rho$

$$= z\hat{i} - 2x\hat{j} + y\hat{k} \,.\, \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\boxed{F_\rho = z\cos\phi - 2\rho\cos\phi\sin\phi} \quad \text{— (2)}$$

$$F_\phi = \vec{F}\cdot\hat{T}_\phi$$

$$= z\hat{i} - 2x\hat{j} + y\hat{k} \,.\, -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\boxed{F_\phi = -z\sin\phi - 2\rho\cos^2\phi} \quad \text{— (3)}$$

$$F_z = \vec{F}\cdot\hat{T}_z$$

$$= z\hat{i} - 2x\hat{j} + y\hat{k} \,.\, \hat{k}$$

$$\boxed{F_z = \rho\sin\phi} \quad \text{— (4)}$$

Substituting eq^n (2) (3) & (4) in (1)

$$\boxed{\vec{F} = z\cos\phi - 2\rho\cos\phi\sin\phi\,\hat{T}_\rho + (-z\sin\phi - 2\rho\cos^2\phi)\hat{T}_\phi + \rho\sin\phi\,\hat{T}_z}$$

(15) Show that spherical coordinate system is orthogonal.

We have to show that

$$\hat{e}_r\cdot\hat{e}_\theta = \hat{e}_\theta\cdot\hat{e}_\phi = \hat{e}_\phi\cdot\hat{e}_r = 0$$

consider,

$$\hat{e}_r\cdot\hat{e}_\theta = (\sin\theta\cos\phi)\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\hat{k} \,.\, (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta)\hat{k}$$

$$= \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta$$

$$= \sin\theta\cos\theta\,(1) - \sin\theta\cos\theta$$

$$= 0$$

$$\hat{e}_\theta\cdot\hat{e}_\phi = (\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\hat{k}) \,.\, -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$= -\cos\theta\cos\phi\sin\phi + \cos\theta\cos\phi\sin\phi$$

$$= 0$$

$$\hat{e}_\phi \cdot \hat{e}_r = (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \cdot (\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$$
$$= -\sin\phi\sin\theta\cos\phi + \cos\phi\sin\theta\sin\phi$$
$$= \underline{\underline{0}}$$

Hence spherical coordinate system is orthogonal.

(16) Show that cylindrical coordinate system is orthogonal

we have to show that

$$\hat{T}_\rho \cdot \hat{T}_\phi = \hat{T}_\phi \cdot \hat{T}_z = \hat{T}_z \cdot \hat{T}_\rho = 0$$

Consider

$$\hat{T}_\rho \cdot \hat{T}_\phi = (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) \cdot (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$
$$= -\cos\phi\sin\phi + \sin\phi\cos\phi$$
$$= 0$$

$$\hat{T}_\phi \cdot \hat{T}_z = (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) \cdot \hat{k}$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$
$$= 0$$

$$\hat{T}_z \cdot \hat{T}_\rho = \hat{k} \cdot (\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$
$$= 0$$

Hence cylindrical coordinate system is orthogonal.

(17) Express $\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$ in spherical coordinate system.

Let $\vec{F} = F_r \cdot \hat{e}_r + F_\theta \cdot \hat{e}_\theta + F_\phi \cdot \hat{e}_\phi$ be the required spherical coordinate system. — (1)

$F_r = \vec{F} \cdot \hat{e}_r$, $F_\theta = \vec{F} \cdot \hat{e}_\theta$, $F_\phi = \vec{F} \cdot \hat{e}_\phi$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

consider,

$$F_r = \vec{F} \cdot \hat{e}_r$$
$$= z\hat{i} - 2x\hat{j} + y\hat{k} \cdot \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$
$$= z\sin\theta\cos\phi - 2x\sin\theta\sin\phi + y\cos\theta$$
$$= r\cos\theta\sin\theta\cos\phi - 2r\sin\theta\cos\phi\sin\theta\sin\phi + r\sin\theta\sin\phi\cos\theta$$

$$\boxed{F_r = r\sin\theta(\cos\theta\cos\phi - 2\cos\phi\sin\theta\sin\phi + \sin\phi\cos\theta)} \quad —(2)$$

$F_\theta = \vec{F}\cdot\hat{e}_\theta$

$= (r\cos\theta\,\hat{i} - 2r\sin\theta\cos\phi\,\hat{j} + r\sin\theta\sin\phi\,\hat{k}) \cdot \cos\phi\cos\theta\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$= r\cos^2\theta\cos\phi - 2r\cos\phi\sin\theta\cos\theta\sin\phi + r\sin\theta\sin\phi\sin\theta$

$$\boxed{F_\theta = r(\cos^2\theta\cos\phi - 2\cos\phi\sin\theta\cos\theta\sin\phi \mp \sin^2\theta\sin\phi)} \quad - (3)$$

$F_\phi = \vec{F}\cdot\hat{e}_\phi$

$= (r\cos\theta\,\hat{i} - 2r\sin\theta\cos\phi\,\hat{j} + r\sin\theta\sin\phi\,\hat{k}) \cdot -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$= -r\cos\theta\sin\phi - 2r\sin\theta\cos^2\phi$

$$\boxed{F_\phi = -r(\cos\theta\sin\phi + 2\sin\theta\cos^2\phi)} \quad - (4)$$

Substitute (2), (3) & (4) in (1)

$\vec{F} = r\sin\theta(\cos\theta\cos\phi - 2\cos\phi\sin\theta\sin\phi + \sin\phi\cos\theta)\hat{e}_r +$
$r(\cos^2\theta\cos\phi - 2\sin\theta\cos\phi - \sin^2\theta\sin\phi)\hat{e}_\theta$
$-r(\cos\theta\sin\phi + 2\sin\theta\cos^2\phi)\hat{e}_\phi$