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SCHUR M - POWER CONVEXITY OF RELATED FUNCTION FOR HOLDERS INEQUALITY WITH APPLICATION

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Abstract. Using majorization inequality theory, we examined the Schur m power convexity of related function for Holders Inequality related function and establishing several applications.

1. Introduction

The set of n-dimensional row vectors on the real number field is assumed throughout this work R^n. Let

$$R^n_+ = \{\nu = (\nu_1, \nu_2, ..., \nu_n) : \nu_t \geq 0, t = 1, 2, ..., n\},$$

By Holder's inequality [11], we have

$$\sum_{l=1}^{n} r_l s_l \leq \left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \tag{1.1}$$

$$\int_r^s \phi(\nu)\psi(\nu)d\nu \leq \left(\int_r^s \big(\phi(\nu)\big)^p d\nu\right)^{\frac{1}{p}} \left(\int_r^s \big(\psi(\nu)\big)^q d\nu\right)^{\frac{1}{q}} \tag{1.2}$$

Here $r_l \geq 0, s_l \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$.

The Schur convexity of functions pertaining to special means is a major research topic that has piqued the interest of a number of mathematicians. There have been countless publications written on this subject in recent years (see [1] , [6]) and their sources. Zhang and Yang ([16], [19]), Chu, Zhang and Wang [3], Shi and Zhang ([13], [14],) , Meng, Chu and Tang [12] , Zheng, Zhang and Zhang [18], Meng, Chu, and Tang citeMeng, and Zheng, Zhang, and Zhang citeZheng investigated the Schur geometrically convex functions and Schur harmonically convex functions as supplements to the Schur convexity of functions. These features of functions have been shown to be beneficial in detecting and demonstrating specific means inequalities (see [8]-[11], [15],[21]-[24]). Dong-Sheng Wang, Chun - Ru Fu† and Huan-Nan Sh [4] investigated the Schur convexity about related function of Holders inequality by using majorization inequality theory .

This paper motivates us to investigate Schur m - power convexity about related function of Holders inequality by using majorization inequality theory.

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2. Preliminaries

To estabilish our main results, we need the following definitions and lemmas.

Definition 2.1. [[1], [2]] .Consider two arbitrary n-tuple elements $\lambda, \mu \in R^n$ $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ and $\mu = (\mu_1, \mu_2, ..., \mu_n) \in R^n$.

(1) For the arrangements of λ and μ in descending order of the form if

$$\sum_{p=1}^{t} \lambda_{[p]} \leq \sum_{p=1}^{t} \mu_{[p]}$$

for $1 \leq t \leq n-1$, λ is said to by majorized by μ, (in icon $\lambda \prec \mu$) and

$$\sum_{p=1}^{n} \lambda_{[p]} = \sum_{p=1}^{n} \mu_{[p]},$$

where $\lambda_{[1]} \geq \cdots \geq \lambda_{[n]}$ and $\mu_{[1]} \geq \cdots \geq \mu_{[n]}$

(2) Let $\Psi \subseteq R^n$ $(n \geq 2)$ $p = 1, 2, \cdots, n$ $\lambda \geq \mu$ means $\lambda_p \geq \mu_p$.
The function $\omega : \Psi \rightarrow R$ is declining if and just if $-\omega$ is escalating.

(3) For $\zeta, \eta \in [0, 1]$ with $\zeta + \eta = 1$, $\Psi \subseteq R^n$ is a convex set, if $(\zeta\lambda_1 + \eta\mu_1, \cdots, \zeta\lambda_n + \eta\mu_n) \in \Psi$ for all λ and μ.

(4) the function $\omega : \Psi \rightarrow R$ is considered to be Schur-convex whenever $\lambda \prec \mu$ on Ψ implies $\omega(\lambda) \leq \omega(\mu)$. ω is Schur concave on Ψ if $-\omega$ is Schur convex.

Lemma 2.2. [7]. *Let $\omega : \Psi \rightarrow R$ be differentiable in Ψ^0 and continuous on Ψ and $\Psi \subseteq R^n$ be symmetric with non-empty interior Ψ^0, then ω is Schur convex on Ψ if and only if ω is symmetric on Ψ and*

$$(p - q)\left(\frac{\partial \omega}{\partial p} - \frac{\partial \omega}{\partial q}\right) \geq 0 (\leq 0) \tag{2.1}$$

Definition 2.3. Let $\phi :$ R $_{++} \rightarrow R$ be defined by

$$\phi(x) = \begin{cases} \dfrac{x^m - 1}{m}, & m \neq 0 \\ \ln x, & m = 0 \end{cases} \tag{2.2}$$

Then a function $f : \Psi \subset R^n_{++} \rightarrow R$ is supposed to be Schur 'm'- power convexity on Ψ if $(\phi(u_1), (\phi(u_2), ...(\phi(u_n)) \prec (\phi(v_1), (\phi(v_2), ...(\phi(v_n)))$ for all $(u_1, u_2.....u_n) \in \Psi$ and $(v_1, v_2.....v_n) \in \Psi$ infers $\phi(u) \leq \phi(v)$.

If $-\phi(u)$ is Schur 'm'- power convex, then we say that $\phi(u)$ is Schur 'm'-power concave.

Lemma 2.4. [20] *Let $\omega : \Psi \rightarrow R$ be differentiable in Ψ^0 and continuous on Ψ and $\Psi \subseteq^n$ be symmetric with non-empty interior Ψ^0, then ω is Schur convex on Ψ If ω is symmetric on Ψ and*

$$\frac{x_1^m - x_2^m}{m}[x_1^{1-m}\frac{\partial \varphi(x)}{\partial x_1} - x_2^{1-m}\frac{\partial \varphi(x)}{\partial x_2}] \geq 0, if \ m \neq 0 \tag{2.3}$$

and

$$(lnx_1 - lnx_2)[x_1\frac{\partial \varphi(x)}{\partial x_1} - x_2\frac{\partial \varphi(x)}{\partial x_2}] \geq 0, if \ m = 0 \tag{2.4}$$

Lemma 2.5. [17].*(Chebyshev's inequality) If progressions $r_n \geq 0$, $s_n \geq 0$. we have*

(1) *When r_n, s_n have opposite monotonicity, then*

$$\sum_{l=1}^{n} r_1 \sum_{l=1}^{n} s_l \geq n \sum_{l=1}^{n} s_l r_l \tag{2.5}$$

(2) *When r_n, s_n have same monotonicity, then*

$$\sum_{l=1}^{n} r_l \sum_{l=l}^{n} s_1 \leq n \sum_{l=1}^{n} s_l r_l \tag{2.6}$$

Lemma 2.6. *If $\phi(\nu)$ is the convex (concave) function on the interval then*

$$\phi\left(\frac{r+s}{2}\right) \leq (\geq) \frac{1}{s-r} \int_{s}^{r} \phi(\nu) d\nu \leq (\geq) (\frac{\phi(r)+\phi(s)}{2}) \tag{2.7}$$

Lemma 2.7. [11]. *Let $\nu = (\nu_1, \nu_2, \nu_3 ... \nu_n) \in R^n_{++}$, then*

$$\underbrace{\left(\frac{1}{(H_n(\nu))}, \frac{1}{(H_n(\nu))}, ..., \frac{1}{(H_n(\nu))}\right)}_{n} \prec \left(\frac{1}{\nu_1}, \frac{1}{\nu_2} ... \frac{1}{\nu_n}\right) = \nu$$

Lemma 2.8. [17]. *(Young's inequality) Suppose $r, s \geq 0, p \geq 1, \frac{1}{p} + \frac{1}{q} = 1$ then*

$$\frac{1}{p} r^p + \frac{1}{q} s^p \geq rs \tag{2.8}$$

Lemma 2.9. *Suppose $r, s \geq 0, p \geq 1, \frac{1}{p} + \frac{1}{q} = 1$ then*

$$rs \leq \frac{1}{p}(r^p + s^p) + \frac{1}{q}(r^q + s^q) - \frac{r^2 + s^2}{2} \tag{2.9}$$

Lemma 2.10. *when $1 \geq r \geq s \geq 0, p \geq q \geq 1$ then*

$$\frac{1}{p} r^p + \frac{1}{q} s^q \leq \frac{1}{p} s^p + \frac{1}{q} r^q \tag{2.10}$$

3. Main Results

In this paper, by using the principle of majorization as an example, combined with majorization inequality, the Schur m power convexity of related function for holder's inequality gives sharpening inequality of the Holders under certain conditions.

Our primary outcome is as follows:

Theorem 3.1. *Let $r_n \geq 0$ and $s_n \geq 0$ be any two progressions and let p and q be two non-zero arbitrary real numbers. Let*

$$H_1(r) = \sum_{l=1}^{n} r_l s_l \leq \left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \tag{3.1}$$

If $p > 0$ and $m < 0$, then $H_1(r)$ is Schur m power convex on R_+ with $r_1, ..., r_n$ and if $p < 0$ and $m > 0$, then $H_1(r)$ is Schur m power concave on R_+ with $r_1, ..., r_n$.

Proof. : Here $H_1(r)$ is obviously symmetric with $r = r_1, ..., r_n$ on R_+.
Let us assume $r_1 > r_2$.
Now by differentiating (3.1) partially with respect to r_1 and r_2, we get

$$\frac{\partial H_1}{\partial r_1} = \left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}-1} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} r_1^{p-1}$$

and

$$\frac{\partial H_1}{\partial r_2} = \left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}-1} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} r_2^{p-1}$$

Consider,

$$\triangle_1 = \frac{r_1^m - r_2^m}{m}\left(r_1^{1-m}\frac{\partial\ H_1}{\partial r_1} - r_2^{1-m}\frac{\partial\ H_1}{\partial r_2}\right)$$

$$\Rightarrow \qquad \triangle_1 = \frac{r_1^m - r_2^m}{m}\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}-1} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \left(r_1^{p-m} - r_2^{p-m}\right)$$

It is easy to see that, when $p > 0$ and $m < 0$, then $\triangle_1 > 0$ and when $p < 0$ and $m > 0$, then $\triangle_1 < 0$.

Hence, by Lemma 2.4,
if $p > 0$ and $m < 0$, then $H_2(s)$ is Schur m power convex on R_+ with $s_1, ..., s_n$ and if $p < 0$ and $m > 0$, then $H_2(s)$ is Schur m power concave on R_+ with $s_1, ..., s_n$.

This completes proof of Theorem 3.1. □

Theorem 3.2. *Let $r_n \geq 0$ and $s_n \geq 0$ be any two progressions and let p and q be two non-zero arbitrary real numbers. Let*

$$H_2(s) = n^{\frac{1}{p}} A_{n,r}\left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \tag{3.2}$$

If $q > 0$ ang $m < 0$, then $H_2(s)$ is Schur m power convex on R_+ with $s_1, ..., s_n$ and if $q < 0$ and $m > 0$,then $H_2(s)$ is Schur m power concave on R_+ with $s_1, ..., s_n$. Here $A_{n,r} = \frac{1}{n}\sum_{l=1}^{n} r_l$.

Proof. : Here $H_2(r)$ is obviously symmetric with $s = s_1, ..., s_n$ on R_+.
Let us assume $s_1 > s_2$.
Now by differentiating (3.2) partially with respect to s_1 and s_2, we get

$$\frac{\partial H_2}{\partial s_1} = n^{\frac{1}{p}} A_{n,r}\left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} s_1^{q-1}$$

and

$$\frac{\partial H_2}{\partial s_2} = n^{\frac{1}{p}} A_{n,r} \left(\sum_{l=1}^{n} s_l^q \right)^{\frac{1}{q}} s_2^{q-1}$$

Consider,

$$\triangle_2 = \frac{s_1^m - s_2^m}{m} \left(s_1^{1-m} \frac{\partial\ H_1}{\partial s_1} - s_2^{1-m} \frac{\partial\ H_1}{\partial r_2} \right)$$

$$\Rightarrow \qquad \triangle_2 = \frac{s_1^m - s_2^m}{m} \left(\sum_{l=1}^{n} r_l^p \right)^{\frac{1}{p}-1} \left(\sum_{l=1}^{n} s_l^q \right)^{\frac{1}{q}} \left(s_1^{q-m} - s_2^{q-m} \right)$$

It is easy to see that, when $q > 0$ and $m < 0$, then $\triangle_2 > 0$ and when $q < 0$ and $m > 0$, then $\triangle_2 < 0$.

Hence, by Lemma 2.4,
if $q > 0$ and $m < 0$, then $H_2(s)$ is Schur m power convex on R_+ with $s_1, ..., s_n$ and if $q < 0$ and $m > 0$, then $H_2(s)$ is Schur m power concave on R_+ with $s_1, ..., s_n$.

This completes proof of Theorem 3.2. □

Theorem 3.3. *Let $\phi(\xi)$ and $\psi(\xi)$ be two continuous functions with $\phi(\xi) > 0$, $\psi(\xi) > 0$ and let $\int_r^s \phi(\xi)\psi(\xi)d\xi \neq 0$, $\int_r^s \left(\phi(\xi)\right)^p d\xi \neq 0$, $\int_r^s \left(\psi(\xi)\right)^q d\xi \neq 0$, where p and q are arbitrary real numbers. Let*

$$H_3\big(r,s\big) = \begin{cases} \left[\dfrac{\int_r^s \left(\psi(\xi)\right)^q d\xi}{\int_r^s \phi(\xi)\psi(\xi)d\xi} \right]^p \left[\dfrac{\int_r^s \left(\phi(\xi)\right)^p d\xi}{\int_r^s \phi(\xi)\psi(\xi)d\xi} \right]^q, & if\ r \neq s \\ \left(\phi(\xi)\psi(\xi)\right)^{pq-p-q}, & if\ r = s \end{cases} \tag{3.3}$$

Then $H_3\big(r,s\big)$ is Schur m power concave(convex) with r, s if and only if:

$$\frac{q\big(\phi^p(s) + \phi^p(r)\big)}{\int_r^s \phi^p(\xi)d\xi} + \frac{p\big(\psi^q(s) + \psi^q(r)\big)}{\int_r^s \psi^q(\xi)d\xi} \leq (\geq) \frac{\Big(\phi(s)\psi(s) + \phi(r)\psi(r)\Big)(p+q)}{\int_r^s \phi(\xi)\psi(\xi)d\xi} \tag{3.4}$$

Proof. : Here $H_3\big(r,s\big)$ is obviously symmetric with $r = r_1, r_2, ..., r_n$ and $s = s_1, s_2, ..., s_n$ on R_+.
Let us assume $s > r$.
From (3.3), we have

$$H_3\big(r,s\big) = \left[\frac{\int_r^s \left(\psi(\xi)\right)^q d\xi}{\int_r^s \phi(\xi)\psi(\xi)d\xi} \right]^p \left[\frac{\int_r^s \left(\phi(\xi)\right)^p d\xi}{\int_r^s \phi(\xi)\psi(\xi)d\xi} \right]^q$$

$$\Rightarrow \quad H_3\big(r,s\big) = \frac{\left(\int_r^s \phi^p(\xi)d\xi\right)^q \left(\int_r^s \psi^q(\xi)d\xi\right)^p}{\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{p+q}}$$

Now by differentiating this partially with respect to s and r, we get

$$\frac{\partial H_3}{\partial s} = \frac{q\left(\int_r^s \phi^p(\xi)d\xi\right)^{q-1} \phi^p(s)\int_r^s \left(\psi^q(\xi)d\xi\right)^p \int_r^s \left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q}}{\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}$$
$$+\frac{p\left(\int_r^s \psi^q(\xi)d\xi\right)^{p-1} \psi^q(s)\int_r^s \left(\phi^q(\xi)d\xi\right)^p \int_r^s \left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q}}{\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}$$
$$-\frac{(p+q)\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{(p+q-1)} \phi(s)\psi(s)\left(\int_r^s \phi^p(\xi)d\xi\right)^q \left(\int_r^s \psi^q(\xi)d\xi\right)^p}{\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}$$

$$\frac{\partial H_3}{\partial r} = \frac{q\left(\int_r^s \phi^p(\xi)d\xi\right)^{q-1} \phi^p(r)\int_r^s \left(\psi^q(\xi)d\xi\right)^p \int_r^s \left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q}}{\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}$$
$$-\frac{p\left(\int_r^s \psi^q(\xi)d\xi\right)^{p-1} \psi^q(r)\int_r^s \left(\phi^q(\xi)d\xi\right)^q \int_r^s \left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q}}{\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}$$
$$+\frac{(p+q)\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{(p+q-1)} \phi(r)\psi(r)\left(\int_r^s \phi^p(\xi)d\xi\right)^q \left(\int_r^s \psi^q(\xi)d\xi\right)^p}{\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}$$

Consider,

$$\triangle_3 = \frac{s^m - r^m}{m}\left(s^{1-m}\frac{\partial\ H_1}{\partial s} - r^{1-m}\frac{\partial\ H_1}{\partial r}\right)$$

This implies that,

$$\triangle_3 = \frac{s^m - r^m}{m\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}\Bigg[q\left(\int_r^s \phi^p(\xi)d\xi\right)^{q-1}\left(\int_r^s \psi^q(\xi)d\xi\right)^p$$

$$\times\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{p+q}\big(s^{1-m}\phi^p(s) + r^{1-m}\phi^p(\xi)\big) + p\left(\int_r^s \psi^q(\xi)d\xi\right)^{p-1}$$

$$\times\left(\int_r^s \phi^p(\xi)d\xi\right)^q\int_r^s\left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q}\left(s^{1-m}\psi^q(s) + r^{1-m}\psi^q(r)\right)$$

$$-(p+q)\int_r^s\left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q-1}\left(\int_r^s \phi^p(\xi)d\xi\right)^q\left(\int_r^s \psi^q(\xi)d\xi\right)^p$$

$$\times\left(s^{1-m}\phi(s)\psi(s) + r1 - m\phi(r)\psi(r)\right)\Bigg]$$

$$= \frac{s^m - r^m}{m\left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{2(p+q)}}\Bigg[q\int_r^s\left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q-1}\left(\int_r^s \phi^p(\xi)d\xi\right)^{q-1}$$

$$\times\left(\int_r^s \psi^q(\xi)d\xi\right)^p\Bigg\{\int_r^s \phi(\xi)\psi(\xi)d\xi\big(s^{1-m}\phi^p(s) + r^{1-m}\phi^p(r)\big)$$

$$-\left(\int_r^s \phi^p(\xi)d\xi\right)\times\left(s^{1-m}\phi(s)\psi(s) + r^{1-m}\phi(r)\psi(r)\right)\Bigg\}$$

$$+p\int_r^s\left(\phi(\xi)\psi(\xi)d\xi\right)^{p+q-1}\left(\int_r^s \phi^p(\xi)d\xi\right)^q$$

$$\times\left(\int_r^s \psi^q(\xi)d\xi\right)^{p-1}\Bigg\{\int_r^s \phi(\xi)\psi(\xi)d\xi\big(s^{1-m}\psi^q(s) + r^{1-m}\psi^p(r)\big)$$

$$-\left(\int_r^s \psi^p(\xi)d\xi\right)\times\left(s^{1-m}\phi(s)\psi(s) + r^{1-m}\phi(r)\psi(r)\right)\Bigg\}\Bigg]$$

$$= \left(\int_r^s \phi(\xi)\psi(\xi)d\xi\right)^{p+q-1}\left(\int_r^s \phi^p(\xi)d\xi\right)^{q-1}\left(\int_r^s \psi^q(\xi)d\xi\right)^{p-1}$$

$$\Big\{q\int_r^s \psi^q(\xi)d\xi\Big[\int_r^s \phi(\xi)\psi(\xi)d\xi\big(s^{1-m}\phi^p(s)+r^{1-m}\phi^p(r)\big)-\int_r^s \phi^p(\xi)d\xi\big(s^{1-m}\phi(s)\psi(s)\big)+r^{1-m}\phi(r)\psi(r)\Big]$$

$$+\,p\int_r^s \phi^p(\xi)d\xi\Big[\int_r^s \phi(\xi)\psi(\xi)d\xi\big(s^{1-m}\psi^p(s)+r\psi^p(r)\big)-\int_r^s \psi^q(\xi)d\xi\big(s^{1-m}\phi(s)\psi(s)\big)+r^{1-m}\phi(r)\psi(r)\Big]\Big\}$$

Since

$$\frac{(s^m-r^m)}{m\Big(\int_r^s \phi(\xi)\psi(\xi)d\xi\Big)^{2(p+q)}}\Big(\int_r^s \phi(\xi)\psi(\xi)d\xi\Big)^{p+q-1}\Big(\int_r^s \phi^p(\xi)d\xi\Big)^{q-1}\Big(\int_r^s \psi^q(\xi)d\xi\Big)^{p-1}\geq 0$$

So $\triangle_3$ and

$$q\int_r^s \psi^q(\xi)d\xi\Big[\int_r^s \phi(\xi)\psi(\xi)d\xi\big(s^{1-m}\phi^p(s)+r^{1-m}\phi^p(r)\big)-\int_r^s \phi^p(\xi)d\xi\big(s^{1-m}\phi(s)\psi(s)\big)+r^{1-m}\phi(r)\psi(r)\Big]$$
$$+\,p\int_r^s \phi^p(\xi)d\xi\Big[\int_r^s \phi(\xi)\psi(\xi)d\xi\big(s^{1-m}\psi^p(s)+r^{1-m}\psi^p(r)\big)-\int_r^s \psi^q(\xi)d\xi\big(s^{1-m}\phi(s)\psi(s)\big)+r^{1-m}\phi(r)\psi(r)\Big]$$
$$=\int_r^s \phi(\xi)\psi(\xi)d\xi\Big[q\int_r^s \psi^q(\xi)d\xi\big(s^{1-m}\phi^p(s)+r^{1-m}\phi^p(r)\big)+p\int_r^s \phi^q(\xi)d\xi\big(s^{1-m}\phi^q(s)+r^{1-m}\psi^q(r)\big)\Big]$$

$$-\int_r^s \phi^p(\xi)d\xi\int_r^s \psi^q(\xi)d\xi\big(s^{1-m}\phi(s)\psi(s)+r^{1-m}\phi(r)\psi(r)\big)\big(p+q\big)$$

have the same symbol.

Hence, we have $H_3\big(r,s\big)$ is Schur m power concave (convex) with r, s, if and only if:

$$\int_r^s \phi(\xi)\psi(\xi)d\xi\Big[q\int_r^s \psi^q(\xi)d\xi\big(s^{1-m}\phi^p(s)+r^{1-m}\phi^p(r)\big)+p\int_r^s \phi^q(\xi)d\xi\big(s^{1-m}\phi^q(s)+r^{1-m}\psi^q(r)\big)\Big]$$

$$\leq(\geq)\int_r^s \phi^p(\xi)d\xi\int_r^s \psi^q(\xi)d\xi\big(s^{1-m}\phi(s)\psi(s)+r^{1-m}\phi(r)\psi(r)\big)\big(p+q\big)$$

$$\Leftrightarrow\ \frac{q\int_r^s \psi^q(\xi)d\xi\big(s^{1-m}\phi^p(s)+r^{1-m}\phi^p(r)\big)+p\int_r^s \phi^q(\xi)d\xi\big(s^{1-m}\phi^q(s)+r^{1-m}\psi^q(r)\big)}{\int_r^s \phi^p(\xi)d\xi\int_r^s \psi^q(\xi)d\xi}$$

$$\leq(\geq)\ \frac{\big(s^{1-m}\phi(s)\psi(s)+r^{1-m}\phi(r)\psi(r)\big)\big(p+q\big)}{\int_r^s \phi(\xi)\psi(\xi)d\xi}$$

$$\Leftrightarrow \frac{q\big(s^{1-m}\phi^p(s)+r^{1-m}\phi^p(r)\big)}{\int_r^s \phi^p(\xi)d\xi}+\frac{p\big(s^{1-m}\psi^q(s)+r^{1-m}\psi^q(r)\big)}{\int_r^s \psi^q(\xi)d\xi}$$

$$\leq (\geq)\frac{\big(s^{1-m}\phi(s)\psi(s)+r^{1-m}\phi(r)\psi(r)\big)(p+q)}{\int_r^s \phi(\xi)\psi(\xi)d\xi}$$

This completes proof of Theorem 3.3.

□

Corollary 3.4. *Let $\phi(\xi)$ and $\psi(\xi)$ be two continuous functions and let their second order derivatives exists with*

$$\phi(x>0),\psi(\xi)>0,\int_r^s \phi(\xi)\psi(\xi)d\xi\neq 0,\int_s^r (\phi(\xi))^p d\xi\neq 0,\int_s^r \psi(\xi)^p d\xi\neq 0.$$

If $-1<p<0, 0<q<1, p+q>0$ and $\phi(\xi),\psi(\xi)$ are concave functions of opposite monotonicity then $H_3(r,s)$ is Schur m power convex with $r=r_1,r_2,...,r_n$, and $s=s_1,s_2,...s_n$ on R_+.

4. Application

The following applications are established by using our main results.

Theorem 4.1. *Let $r_n\geq 0$ and $s_n\geq 0$ be any two progressions and let p and q be two non-zero arbitrary real numbers. Then*

(i) *if $p>0, q>0$ then*

$$\Big(\sum_{i=1}^n r_l^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n r_l^q\Big)^{\frac{1}{q}}\geq \Big(n^{\frac{1}{p}}+\frac{1}{q}\Big)A_{n,r}\,A_{n,s}\,.$$

(ii) *if $p<0, q<0$ then*

$$\Big(\sum_{i=1}^n r_l^p\Big)^{\frac{1}{p}}\Big(\sum_{i=1}^n r_l^q\Big)^{\frac{1}{q}}\leq \Big(n^{\frac{1}{p}}+\frac{1}{q}\Big)A_{n,r}\,A_{n,s}\,.$$

Here

$$A_{n,r}=\frac{\sum_{l=1}^n(r_l)}{n},$$
$$A_{n,s}=\frac{\sum_{l=1}^n(s_l)}{n}$$

Proof. : (i) By Lemma 2.7 has a majorization inequality:

$$\Big(\frac{1}{r_1},\frac{1}{r_2},...,\frac{1}{r_n}\Big)\prec\Big(\frac{\frac{1}{r_1}+\frac{1}{r_2}+...+\frac{1}{r_n}}{n},...,\frac{\frac{1}{r_1}+\frac{1}{r_2}+...+\frac{1}{r_n}}{n}\Big)$$

and by Theorem 3.1 and by Ddefinition 2.1, we have if $p > 0$, then $H_1(r) \geq H_1(A_n, r)$, that is

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \geq \left(n(A_{n,r})^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} = n^{\frac{1}{p}} A_{n,r} \left(n(A_{n,s})^q\right)^{\frac{1}{q}}$$

By majorization inequality, we have

$$\left(\frac{1}{s_1}, \frac{1}{s_2}, ..., \frac{1}{s_n}\right) \prec \left(\frac{\frac{1}{s_1} + \frac{1}{s_2} + ... + \frac{1}{s_n}}{n}, ..., \frac{\frac{1}{s_1} + \frac{1}{s_2} + ... + \frac{1}{r_n}}{n}\right)$$

and by Theorem 3.2 and Definition 2.1, we have
if $q > 0$, then $H_2(s) \geq H_2(A_n, s)$, that is

$$n^{\frac{1}{p}} A_{n,r} \left(\sum_{l=1}^{n} s_j^q\right)^{\frac{1}{q}} \geq n^{\frac{1}{p}} A_{n,r} \left(n(A_{n,s})^q\right)^{\frac{1}{q}} = n^{\left(\frac{1}{p}+\frac{1}{q}\right)} A_{n,r} \ A_{n,s}$$

From the above relations, we have

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_j^q\right)^{\frac{1}{q}} \geq n^{\left(\frac{1}{p}+\frac{1}{q}\right)} A_{n,r} A_{n,s}$$

exactness.
By Similar method the following inequality is also established,

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_j^q\right)^{\frac{1}{q}} \leq n^{\left(\frac{1}{p}+\frac{1}{q}\right)} A_{n,r} A_{n,s} \tag{4.1}$$

The proof of Theorem 4.1 is complete. □

Theorem 4.2. *Let $r_n \geq 0$ and $s_n \geq 0$ be any two progressions and let p and q be two non-zero arbitrary real numbers . Then*

(i) *When $p > 0$, if $\frac{1}{p} + \frac{1}{q} = 1$ and $\{r_n\}$,$\{s_n\}$ have the opposite of monotonicity, then*

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \geq n \ A_{n,r} A_{n,s} \geq \sum_{l=1}^{n} r_l s_l$$

(ii) *When $0 < p < 1$, if $\frac{1}{p} + \frac{1}{q} = 1$ and $\{r_n\}$,$\{s_n\}$ have the opposite of monotonicity, then*

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \leq n \ A_{n,r} A_{n,s} \geq \sum_{l=1}^{n} r_l s_l$$

Proof. : (i) When $p > 1$, if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and by Theorem 1, we have

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \geq n\ A_{n,r}\, A_{n,s} = A_{n,r}\, A_{n,s}$$

and by Lemma 2.7, we have

$$n\ A_{n,r}\, A_{n,s} = n\frac{\sum_{l=1}^{n}(r_l)}{n}\frac{\sum_{l=1}^{n}(s_l)}{n} = \frac{\sum_{l=1}^{n}(r_l)\sum_{l=1}^{n}(s_l)}{n} \geq n\ \frac{\sum_{l=1}^{n} r_l s_l}{n} = \sum_{l=1}^{n} r_l s_l$$

From the above relations, we have

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \geq nA_{n,r}\, A_{n,s} \geq \sum_{l=1}^{n} r_l s_l$$

exactness.
By Similar method the following inequality is also established

$$\left(\sum_{l=1}^{n} r_l^p\right)^{\frac{1}{p}} \left(\sum_{l=1}^{n} s_l^q\right)^{\frac{1}{q}} \leq nA_{n,r}\, A_{n,s} \geq \sum_{l=1}^{n} r_l s_l$$

The proof of Theorem 4.2 is complete.

5. Conclusion

In this paper, by using of majorization inequality theory we investigated the Schur m power convex about related functions of Holder's Inequality, giving a complete critical condition of Schur m power convex function to Holder's Inequality and some applications were established.

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